4.6 – Dimension

Theorem 4.6.2 Let V be a finite-dimensional vector space, and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be any basis for V.
a) If a set in V has more than n vectors, then it is linearly dependent.
b) If a set in V has fewer than n vectors, then it does not span V.

The proof of this essentially involves counting variables and equations in a linear system.

Theorem 4.6.1 All bases for a finite-dimensional vector space have the same number of vectors.

Definition: The **dimension** of a finite-dimensional vector space V is denoted by $\dim(V)$ and is defined to be the number of vectors in a basis for V (in some physical contexts, "dimension" is referred to as **degrees of freedom**). In addition, the zero vector space is defined to have dimension zero.

(Recall that the basis for $\{\vec{0}\}$ is $\emptyset$.

Find a basis for the solution space of each homogeneous linear system, and find the dimension of that space.

#1
$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ -2x_1 - x_2 + 2x_3 &= 0 \\ -x_1 \phantom{{}- x_2} + x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ -2 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1: x_1 = x_3$, $R_2: x_2 = 0$. Let $x_3 = t$

$$\vec{x} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} t$$ Solution space is span$\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$.

The basis of the solution space is $\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$.

Dimension is 1 (# free variables).

#5 $2x_1 - 6x_2 + 2x_3 = 0$

$3x_1 - 9x_2 + 3x_3 = 0$

$$\begin{bmatrix} 2 & -6 & 2 \\ 3 & -9 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 3x_2 - x_3$. Let $s = x_2$, $t = x_3$

$$\vec{x} = \begin{bmatrix} 3s - t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} t$$

Solution space is span $\left\{\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$.

Basis is $\left\{\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right\}$.

Dimension of solution space is 2

#8 In each part, find a basis for the given subspace of R^4, and state its dimension.

a. All vectors of the form $(a, b, c, 0)$.

b. All vectors of the form (a, b, c, d), where $d = a + b$ and $c = a - b$.

c. All vectors of the form (a, b, c, d), where $a = b = c = d$.

a) $\{(e, 0, 0, 0), (0, 100, 0, 0), (0, 0, 2, 0)\}$ dim : 3

b) $$\begin{bmatrix} a \\ b \\ a-b \\ a+b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} a + \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} b$$ Basis: $\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right\}$.

dim : 2

c) basis: $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ a \\ a \\ a \end{bmatrix} \rightarrow$ basis: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

dim: 1

- correct # vectors
- lin. indep.
- span

} any two of these guarantees the third.

Theorem 4.6.4 Let V be an n-dimensional vector space, and let S be a set in V with exactly n vectors. Then S is a basis for V if and only if S spans V or S is linearly independent.

$\rightleftarrows$

pf: ($\Rightarrow$) If S is a basis for V, then S spans V and S is independent by def.

($\Leftarrow$) Suppose S has exactly n vectors and spans V. If S is not linearly independent, then there is $\vec{v}_j$ in S such that

$\vec{v}_j = \sum_{i=1}^{n} k_i \vec{v}_i$, $i \neq j$. Then $\vec{v}_j$ can be removed

without affecting the span, contradicting that V is n-dimensional. So S is independent. Thus, S is a basis for V.

Suppose S has exactly n vectors and is linearly independent. If S does not

span V then $\exists\, \vec{v} \in V \ni \vec{v} \notin \text{span}(S)$.
Then $S \cup \{\vec{v}\}$ is independent, again contradicting that V is n-dimensional. So S spans $V \Rightarrow S$ is a basis for V.

Theorem 4.6.5 Let S be a finite set of vectors in a finite-dimensional vector space V.
a) If S spans V but is not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S.
b) If S is a linearly independent set that is not already a basis for V, then S can be enlarged to a basis for V by inserting appropriate vectors into S.

#12 Find standard basis vectors for R^3 that can be added to the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to produce a basis for R^3.
a. $\mathbf{v}_1 = (-1, 2, 3)$, $\mathbf{v}_2 = (1, -2, -2)$
b. $\mathbf{v}_1 = (1, -1, 0)$, $\mathbf{v}_2 = (3, 1, -2)$

$R_2 \to R_2 + 2R_1$
$$\begin{array}{rrrrr} 2 & -2 & 0 & 1 & 0 \\ -2 & 2 & 2 & 0 & 0 \\ \hline 0 & 0 & 2 & 1 & 0 \end{array}$$

$R_3 \to R_3 + 3R_1$
$$\begin{array}{rrrrr} 3 & -2 & 0 & 0 & 1 \\ -3 & 3 & 3 & 0 & 0 \\ \hline 0 & 1 & 3 & 0 & 1 \end{array}$$

a)
$$\begin{array}{ccccc} \vec{v}_1 & \vec{v}_2 & \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{array}$$
$$\left[\begin{array}{rr|rrr} -1 & 1 & 1 & 0 & 0 \\ 2 & -2 & 0 & 1 & 0 \\ 3 & -2 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rr|rrr} -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 1 & 3 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{rr|rrr} -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 2 & 1 & 0 \end{array}\right]$$

We can add $\vec{e}_1$.

Pivot columns indicate basis vectors

(Note: $\vec{e}_2$ also works)

$$\begin{array}{ccccc} \vec{v}_1 & \vec{v}_2 & \vec{e}_3 & \vec{e}_2 & \vec{e}_1 \end{array}$$
$$\left[\begin{array}{rr|rrr} -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 2 & 1 \end{array}\right]$$

b) $\left[\begin{array}{cc|ccc} 1 & 3 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|ccc} 1 & 3 & 1 & 0 & 0 \\ 0 & 4 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|ccc} 1 & 3 & 1 & 0 & 0 \\ 0 & 4 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right]$

$R_2 \Rightarrow R_1 + R_2$

$$\begin{array}{ccccc} 1 & 3 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ \hline 0 & 4 & 1 & 1 & 0 \end{array}$$

$R_3 \Rightarrow R_2 + 2R_3$

$$\begin{array}{ccccc} 0 & 4 & 1 & 1 & 0 \\ 0 & -4 & 0 & 0 & 2 \\ \hline 0 & 0 & 1 & 1 & 2 \end{array}$$

#18 Find a basis for the subspace of R^4 that is spanned by the vectors $\mathbf{v}_1 = (1, 1, 1, 1)$, $\mathbf{v}_2 = (2, 2, 2, 0)$, $\mathbf{v}_3 = (0, 0, 0, 3)$, $\mathbf{v}_4 = (3, 3, 3, 4)$.

$$\begin{array}{cccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{array}$$
$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 1 & 2 & 0 & 3 \\ 1 & 2 & 0 & 3 \\ 1 & 0 & 3 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & -3 & -1 \end{bmatrix}$$

$\vec{c}_1, \vec{c}_2$ are pivot columns

The desired basis is $\{(1,1,1,1), (2,2,2,0)\}$.

Note: $\vec{v}_3 = 3\vec{v}_1 - \frac{3}{2}\vec{v}_2$, $\vec{v}_4 = 4\vec{v}_1 - \frac{1}{2}\vec{v}_2$

This is not a unique answer.

The subspace spanned by the set has 2 dimensions.

$(1,-1), (0,4), (2,0), (-1,0)$ $A\vec{x} = \vec{0}$

#20 In each part, let T_A be multiplication by A and find the dimension of the subspace of R^4 consisting of all vectors **x** for which $T_A(\mathbf{x}) = \mathbf{0}$.

a. $\begin{bmatrix} 1 & 0 & 2 & -1 \\ -1 & 4 & 0 & 0 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

That is, find the dimension of the kernel.

a) $\begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 4 & 2 & -1 \end{bmatrix}$ 4 var. 2 leading 1s $\rightarrow$ 2 free vars

dim: 2

$x_1 = -2x_3 + x_4$

$x_2 = -\frac{1}{2}x_3 + \frac{1}{4}x_4$

$\vec{x} = \begin{bmatrix} -2 \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 1 \\ 1/4 \\ 0 \\ 1 \end{bmatrix} t$

b) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ 1 free variable means the dimension is 1.

Note: This is the dimension of $\ker(T_A)$.